ON THE STEADY VISCOUS FLOW OF A NONHOMOGENEOUS ASYMMETRIC FLUID

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Abstract. We consider a boundary value problem for the system of equations describing the stationary motion of a viscous nonhomogeneous asymmetric fluid in a bounded planar domain having a C^2 boundary. We use a stream-function formulation after the manner of N. N. Frolov [Math. Notes, **53**(5-6), 650–656, 1993] in which the fluid density depends on the stream-function by means of another function determined by the boundary conditions. This allows for dropping some of the equations, most notably the continuity equation. With a fixed point argument we show the existence of solutions to the resulting system.

Introduction

Density dependent fluids are well known and have been studied by several authors. Antontsev, Kazhikov and Monakhov treat in [1] an assortment of problems on density dependent flows of either compressible or incompressible Newtonian fluids. A more recent account on such problems and some improvements on results in [1] are available in [10] and references therein.

The case of non-Newtonian fluids is less studied then the previous ones. Several constitutive laws lead to such fluids and, among them, we are particularly interested in the non-symmetric fluids, named micropolar fluids, introduced in [4]. Particles of these fluids undergo translations and rotations as well and their theory has proved to be useful in describing phenomena in which the structure of the fluid should be accounted for, *e.g.*, blood flow in thin vessels or flows of some slurries and polymeric fluids, see [11].

Some authors studied evolutionary density dependent flows of micropolar fluids, such as the papers [2, 3, 6] whereas, to the best of our knowledge, basic results on flows in a stationary regime are still lacking. Our aim in this paper is, therefore, to extend the result in [5] placing the theory of micropolar fluids in a similar level of knowledge as the theory of the standard Navier-Stokes fluids.

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Frolov addressed in [5] the stationary 2D flow of an incompressible inhomogeneous fluid resorting to a judicious stream-function formulation. His formulation has been successfully adapted to solve new problems as the mixing of two fluids having different (and discontinuous) densities [13] and to the boundary control of steady state flow of a viscous incompressible nonhomogeneous fluid [8].

The field equations of the model, in a steady state regime, form the following system

$$
\begin{aligned}
-(\mu+\mu_r)\Delta\vec{v}+\rho(\vec{v}\cdot\nabla)\vec{v}+\nabla p &= 2\mu_r\nabla\times\vec{w}+\rho\vec{f}\\
\nabla\cdot(\rho\vec{v})=0, \qquad \nabla\cdot\vec{v}&=0,\\
-(c_a+c_d)\Delta\vec{w}+\rho\jmath(\vec{v}\cdot\nabla)\vec{w}-(c_0-c_a+c_d)\nabla(\nabla\cdot\vec{w})+4\mu_r\vec{w} &= 2\mu_r\nabla\times\vec{v}+\rho\vec{g}
\end{aligned}
\tag{1}
$$

which we assume to hold in a bounded planar domain Ω, having a C^2 boundary $\partial\Omega$, subject to the following boundary conditions

$$
\rho=\rho_0>0 \quad \text{on } \Gamma, \qquad \vec{v}=\vec{v}_0,\ \vec{w}=\vec{w}_0 \quad \text{on } \partial\Omega, \quad \text{with } \int_{\partial\Omega}\vec{v}_0\cdot\vec{n}\,ds=0 \tag{2}
$$

with $\Gamma\subset\partial\Omega$ being a connected arc on which $\vec{v}_0\cdot\vec{n}<0$, that is, Γ is the part of $\partial\Omega$ where the fluid flows inward.

The equations (1) represent conservation of linear momentum, the continuity equation, the incompressibility and the conservation of angular momentum respectively. The unknowns are ρ, the denstity; $\vec{v}$ and $\vec{w}$, the fields of velocity and rotation of particles and the pressure, p. The fields $\vec{f}$ and $\vec{g}$ are, respectively, given external sources of linear and angular momenta densities whereas $\mu,\mu_r,c_0,c_a,c_d,\jmath$ are positive constants characterizing the medium and also satisfying $c_0>c_a+c_d$.

Notice that this model contains the incompressible, density dependent Navier-Stokes system as a particular case ($\mu_r=0,\vec{w}\equiv 0$), and it is named the nonhomogeneous micropolar fluid model. More details on the derivation of the model as well as the physical meaning of the several parameters above may be found in [11, 4].

Below we devised a way to solve problem (1)-(2) by combining arguments used in [11, 5]. This is a new result in theory and, although its proof follows fairly well-grooved lines, we believe it may serve as a first step towards a better understanding of similar flows in different geometries such as infinite or semi-infinite pipes.

Begining with Frolov's stream-function approach allows us to drop the continuity equation and the boundary condition on ρ, cf. Section 2. Then we consider an auxiliary linear problem, for the rotational velocity $\vec{w}$, which we solve with the Lax-Milgram lemma, see Section 3.1. Next we solve a problem for the translational velocity, $\vec{v}$, by defining a suitable operator and using the Leray-Schauder principle, cf. Section 3.2. Thus we get a pair $\vec{v},\vec{w}$ as a weak solution to our problem. We conclude our paper with some remarks concerning the solution so obtained.

1. Notations and statement of the main result

We shall now introduce some notations and clarify what is meant by (1) to hold in a planar domain. Given $\vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, v_3)$, we denote $[(\vec{u} \cdot \nabla)\vec{v}]_j = \vec{u} \cdot \nabla v_j = \sum_k u_k \partial_{x_k} v_j$, $j = 1, 2, 3$. Loosely speaking, a planar flow may be sought as a "slice" of a 3D one, that is, the flow takes place in a cross-section $x_3 =$ const. of a 3D domain. Thus, the functions involved are assumed to be independent of the x_3 variable and the axes of rotation of the particles of the fluid are assumed to be perpendicular to the plane of the flow. This way, we regard

$$\vec{v} = (v_1(x_1, x_2), v_2(x_1, x_2), 0), p = p(x_1, x_2), \vec{w} = (0, 0, w_3(x_1, x_2)),$$
$$\vec{f} = (f_1(x_1, x_2), f_2(x_1, x_2), 0), \vec{g} = (0, 0, g_3(x_1, x_2)),$$

and write $\nabla^\perp \psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$ and $\nabla \times (\phi_1, \phi_2) = \partial_{x_1}\phi_2 - \partial_{x_2}\phi_1$ so that the system (1) may be written componentwise as

$$\begin{gathered} -(\mu + \mu_r)\Delta \vec{v}_j + \rho \vec{v} \cdot \nabla v_j + \partial_{x_j} p = (-1)^{j-1} 2\mu_r \partial_{x_j} w_3 + \rho f_j, \ j = 1, 2 \\ \nabla \cdot (\rho \vec{v}) = 0, \ \nabla \cdot \vec{v} = 0 \\ -(c_a + c_d)\Delta w_3 + \rho \jmath \vec{v} \cdot \nabla w_3 + 4\mu_r w_3 = 2\mu_r \nabla \times \vec{v} + \rho g_3, \ \text{in } \Omega. \end{gathered} \tag{3}$$

From now on, we shall adopt $\jmath \equiv 1$, $\sigma = \mu + \mu_r$ and $\kappa = c_a + c_d$ to shorten the equations.

We use standard notations regarding Sobolev spaces modelled in $L^q(\Omega)$, $W^{k,q}(\Omega), k \geq 0, q > 1$, and their norms $\|\cdot\|_{W^{k,q}}$. The same goes to the trace spaces, $W^{k-1/q,q}(\partial\Omega), k \geq 0, q > 1$, and their norms $\|\cdot\|_{W^{k-1/q,q}}$. For $q = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$ and $W^{1/2,2}(\partial\Omega) = H^{1/2}(\partial\Omega)$, $k \geq 0$, as usual.

By $\mathcal{V}$ we denote the set of divergence-free vector fields $\varphi = (\varphi_1, \varphi_2)$ such that $\varphi \in C_0^\infty(\Omega)$, V is the closure of $\mathcal{V}$ in the H^1-norm and $\mathbf{H} = \{\varphi \in H^1 \mid \nabla \cdot \varphi = 0 \text{ in } \Omega\}$. By $C^{m,\beta}(\Omega)$ we denote the set of all m times continuously differentiable functions in Ω whose m-th order derivatives are Hölder continuous with exponent $\beta \in (0,1)$. As above $\|\cdot\|_{C^\beta}$ stands for the $C^{0,\beta}(\Omega)$-norm.

Our main result then reads

Theorem 1. *Let $\vec{f}, \vec{g} \in L^2(\Omega)$, $\rho_0 \in C^{0,\beta}(\Gamma)$, $0 < \beta < 1$, and $\vec{v}_0, \vec{w}_0 \in H^{1/2}(\partial\Omega)$ be given satisfying* (2). *There exists a weak solution $\rho \in C^{0,\alpha}(\overline{\Omega}), \alpha < \beta, \vec{v} \in \mathbf{H}$, $\vec{w} \in H^1(\Omega)$, of system* (3) *in the sense of Definition 1 below, provided that μ, κ are sufficiently large so that* $\min\{\mu, 2\kappa\} > C\|\eta\|_{L^\infty}\|\vec{w}_0\|_{H^{1/2}}$.

2. Weak formulation of the problem

For a given a divergence-free vector field $\vec{v} = (v_1, v_2)$ in Ω there exists $\phi : \Omega \to \mathbb{R}$ such that $\vec{v} = \nabla^\perp \phi$. In addition, denoting by $\vec{\tau}, \vec{n}$ the unit tangent and outward normal fields

on $\partial\Omega$ and bearing in mind conditions (2), the assumption $\nabla^{\perp}\phi = \vec{v}_0$ on Γ amounts to

$$\frac{\partial \phi}{\partial \vec{n}} = \vec{v}_0 \cdot \vec{\tau}, \qquad \frac{\partial \phi}{\partial \vec{\tau}} = -(\vec{v}_0 \cdot \vec{n}), \quad x \in \Gamma.$$

Thus, the boundary values of ϕ may be obtained upon integration, with respect to the arc length, from a point $\overline{x} \in \Gamma$, $\phi(x) = -\int_{\overline{x}}^{x} \vec{v}_0 \cdot \vec{n}\, ds$, $x \in \Gamma$. Moreover $\phi \in H^{3/2}(\Gamma) \subset C(\Gamma)$, as $\vec{v}_0 \cdot \vec{n} \in H^{1/2}(\Gamma)$. From $\vec{v}_0 \cdot \vec{n} < 0$ on Γ we see that ϕ is increasing on Γ and we may speak of its inverse $\phi^{-1} : \phi(\Gamma) \subset \mathbb{R} \to \Gamma$. We may then define $\tilde{\eta}(y) = \rho_0(\phi^{-1}(y))$, $y \in \phi(\Gamma) \subset \mathbb{R}$ and extend it to $\mathbb{R}$ as a strictly positive function $\eta \in C^{0,\beta}(\mathbb{R}), \beta < 1$, such that $\eta(\psi(x)) = \rho_0(x), x \in \Gamma$, whenever $\nabla^{\perp}\psi = \vec{v}_0$ on Γ.

We fix this η meeting the above requirements. For sufficiently smooth η, ψ we have $\nabla \cdot [\eta(\psi)\nabla^{\perp}\psi] = \eta'(\psi)\nabla\psi \cdot \nabla^{\perp}\psi \equiv 0$, in Ω. A weak version of it may be proved by regularizing η if its assumed to be merely continuous. From all the above facts, we assume that the density has the form $\rho = \eta(\psi)$, where $\nabla^{\perp}\psi = \vec{v}$, whence, the continuity equation $(3)_2$ may be dropped as well as the boundary condition $(2)_1$ on ρ, see [5].

We follow [8] and denote by $N : \mathbf{H} \to H^2(\Omega)$ the continuous operator assigning to each divergence-free vector field, $\vec{u}$, in Ω its stream-function $\psi = N\vec{u}$. Actually such a stream-function is determined up to an arbitrary additive constant which we take to be zero with no loss of generality, see e.g. [14, Lemma 2.5, Chapter 1] or [7, Theorem 4].

Let $\vec{v}_0, \vec{w}_0 \in H^{1/2}(\partial\Omega)$ satisfy $(2)_3$. From the trace theorem follows the existence of $b \in H^1(\Omega)$ with

$$b|_{\partial\Omega} = \vec{w}_0 \text{ and the estimate } \|b\|_{H^1} \leq C\|\vec{w}_0\|_{H^{1/2}}, \tag{4}$$

holds for some absolute constant C. Also, for each $\delta > 0$ fixed, there exists the so-called Leray-Hopf extension of $\vec{v}_0$, that is, a vector field $\vec{a} \in H^2(\Omega)$ satisfying,

$$\begin{aligned} &\vec{a}|_{\partial\Omega} = \vec{v}_0, \qquad \int_{\Omega} \vec{a}^2 \cdot \varphi^2\, dx \leq \delta^2 \int_{\Omega} |\nabla\varphi|^2\, dx, \qquad \varphi \in V \\ &\nabla \cdot \vec{a} = 0, \text{ in } \Omega, \qquad \vec{a}(x) = 0, \quad d(x, \partial\Omega) > \varepsilon, \text{ for a fixed } \varepsilon > 0. \end{aligned} \tag{5}$$

For a construction of the above cut-off functions, the reader is referred to [9]. From this point on we shall omit the domain of integration since no boundary integrals appear in the subsequent computations.

Next we introduce new unknowns $\vec{u} = \vec{v} - \vec{a}$, $\underline{w} = w - b$, $\vec{u} \in V$ and $\underline{w} \in H_0^1(\Omega)$, satisfying the following system of equations in Ω

$$
\begin{aligned}
-\sigma\Delta\vec{u} + \rho[(\vec{u}\cdot\nabla)\vec{u} + (\vec{a}\cdot\nabla)\vec{u} + (\vec{u}\cdot\nabla)\vec{a}] + \nabla p &= -2\mu_r\nabla^{\perp}\underline{w} + \rho\vec{f} + \tilde{F}\\
\nabla\cdot(\rho\vec{u}) = 0, \qquad \nabla\cdot\vec{u} &= 0,\\
-\kappa\Delta\underline{w} + \rho[\vec{u}\cdot\nabla\underline{w} + \vec{u}\cdot\nabla b + \vec{a}\cdot\nabla\underline{w}] + 4\mu_r\underline{w} &= 2\mu_r\nabla\times\vec{u} + \rho\vec{g} + \tilde{G},\\
\vec{u}, \underline{w} = 0, \text{ on } \partial\Omega&,
\end{aligned}
\tag{6}
$$

with $\tilde{F} = \tilde{F}(\rho, \vec{a}, b) := \sigma\Delta\vec{a} - \rho(\vec{a}\cdot\nabla)\vec{a} - 2\mu_r\nabla^{\perp}b$ and $\tilde{G} = \tilde{G}(\rho, \vec{a}, b) := \kappa\Delta b - \rho(\vec{a}\cdot\nabla)b - 4\mu_r b + 2\mu_r\nabla\times\vec{a}$.

To solve the boundary value problem (6) we look for $\rho = \eta(N(u + a))$ and set $F = F(\vec{u}, \vec{a}, b) := \tilde{F}(\eta(N[\vec{u} + \vec{a}]), \vec{a}, b)$ and $G = G(\vec{u}, \vec{a}, b) := \tilde{G}(\eta(N[\vec{u} + \vec{a}], \vec{a}, b)$. From the imbedding $H^2(\Omega) \subset C^{0,\theta}(\overline{\Omega}), \theta \in (0, 1)$, and $\eta \in C^{0,\beta}(\mathbb{R})$ we see $\rho \in C^{0,\alpha}(\overline{\Omega})$, for $\alpha = \beta\theta < \beta$.

Definition 1. *We call a pair $\vec{u} \in V, \underline{w} \in H_0^1(\Omega)$ a weak solution of system* (6) *if the integral identities*

$$
\begin{aligned}
\sigma\int\nabla\vec{u}\cdot\nabla\varphi\,dx &= \int\eta(N(\vec{u}+\vec{a}))[\vec{u}\cdot\nabla\varphi\cdot\vec{u} + \vec{a}\cdot\nabla\varphi\cdot\vec{u} + \vec{u}\cdot\nabla\varphi\cdot\vec{a}]\,dx\\
&\quad - 2\mu_r\int\nabla^{\perp}\underline{w}\cdot\varphi\,dx + \int\eta(N(\vec{u}+\vec{a}))\vec{f}\cdot\varphi\,dx + \int F\varphi\,dx\\
\kappa\int\nabla\underline{w}\nabla\xi\,dx &= \int\eta(N(\vec{u}+\vec{a}))[(\vec{u}\cdot\nabla\xi)\underline{w} + (\vec{u}\cdot\nabla\xi)b + (\vec{a}\cdot\nabla\xi)\underline{w}]\,dx\\
&\quad - 2\mu_r\int(2\underline{w} - \nabla\times\vec{u})\xi\,dx + \int\eta(N(\vec{u}+\vec{a}))g\xi\,dx + \int G\xi\,dx
\end{aligned}
\tag{7}
$$

hold for all $\varphi \in V, \xi \in H_0^1(\Omega)$ and F, G as above.

It is readily seen that a weak solution $\vec{u}, \underline{w}$ of system (6) in the sense of the above definition yields a weak solution $\vec{v} = \vec{u} + \vec{a}, w = \underline{w} + b$ to the original problem (1)-(2). Indeed, recalling (4) and (5), we see that the boundary conditions $(2)_2$ hold in the sense of traces. Moreover the integral identities (7) for $\vec{u} = \vec{v} - \vec{a}$, $\underline{w} = w - b$ imply analogous ones for $\vec{v}, w$.

It is worth remarking that the recovery of the pressure is a standard matter and it follows from the knowledge of $\rho, \vec{v}, w$, see e.g. [14]. To be precise, we state it below as

Theorem 2. *Let $\rho \in C^{0,\beta}(\overline{\Omega}), 0 < \beta < 1, \vec{v} \in \mathbf{H}, w \in H^1(\Omega)$ be a weak solution of the problem* (1)-(2). *Then there is a $p \in L^2(\Omega)$, such that $\nabla p \in L^2(\Omega)$ and for all $\psi \in C_0^{\infty}(\Omega)$ it holds that*

$$
\sigma\int\nabla\vec{v}\cdot\nabla\psi\,dx - \int\rho\big((\vec{v}\cdot\nabla)\psi\cdot\vec{v} - \vec{f}\cdot\psi\big)\,dx + 2\mu_r\int\nabla^{\perp}w\cdot\psi\,dx = \int p\nabla\cdot\psi\,dx.
$$

3. Proof of Theorem 1

We split this proof in three steps.

3.1. **Auxiliary problem.**

In this section we consider the following auxiliary problem: (A) given $\vec{v} \in \mathbf{H}$, find $\underline{w} \in H_0^1(\Omega)$ such that the identity

$$-\kappa\Delta\underline{w} + \eta(N\vec{v})\vec{v} \cdot \nabla\underline{w} + 4\mu_r\underline{w} = 2\mu_r\nabla \times \vec{v} + \eta(N\vec{v})g + G(\vec{v}, b), \text{ in } \Omega$$

holds in the sense of distributions, with $G(\vec{v}, b) = \kappa\Delta b - \eta(N\vec{v})\vec{v} \cdot \nabla b - 4\mu_r b$. Existence of an unique $\underline{w} \in H_0^1(\Omega)$ solving problem (A) follows from the Lax-Milgram lemma. In fact, the problem (A) amounts to find $\underline{w} \in H_0^1(\Omega)$ such that

$$B_v[\underline{w}, \xi] = \int \Big(2\mu_r\nabla \times \vec{v} + \eta(N\vec{v})g + G(\vec{v}, b)\Big)\xi \, dx, \quad \text{for all } \xi \in H_0^1(\Omega) \tag{8}$$

and $B_v : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ being defined as

$$B_v[\chi, \xi] = \int \left[\kappa\nabla\chi \cdot \nabla\xi + \eta(N\vec{v})\vec{v} \cdot \nabla\chi\xi + 4\mu_r\chi\xi\right] dx, \quad \text{for all } \chi, \xi \in H_0^1(\Omega).$$

According to standard estimates B_v is readily seen to be continuous.

It is also coercive as for all $\vec{v} \in \mathbf{H}$, $\underline{w} \in H_0^1(\Omega)$ we have $\int \eta(N\vec{v})\vec{v} \cdot \nabla\underline{w}\, \underline{w} \, dx = 0$, whence

$$B_v[\underline{w}, \underline{w}] = \kappa\|\nabla\underline{w}\|_{L^2}^2 + 4\mu_r\|\underline{w}\|_{L^2}^2 \geq \min\{\kappa, 4\mu_r\}\|\underline{w}\|_{H^1}^2,$$

for all $\underline{w} \in H_0^1(\Omega)$.

In addition, the right-hand-side of (8) is a continuous form in $H_0^1(\Omega)$,

$$\left|\int \Big(2\mu_r\nabla \times v + \eta(N\vec{v})g + G(\vec{v}, b)\Big)\xi \, dx\right| \leq C\|\xi\|_{H^1},$$

for all $\xi \in H_0^1(\Omega)$, with a constant C depending on $c_a, c_d, \mu_r, \|\eta\|_{L^\infty}, \|g\|_{L^2}, \|\vec{w}_0\|_{H^{1/2}}$, $\|\vec{v}\|_{H^1}$ and Ω. The Lax-Milgram lemma assures the existence of an unique $\underline{w} \in H_0^1(\Omega)$ solving problem (A).

We point out for future reference that the following estimate holds for $\underline{w}$:

$$\begin{aligned} \kappa\|\nabla\underline{w}\|_{L^2}^2 + 4\mu_r\|\underline{w}\|_{L^2}^2 &\leq 2\mu_r\|\nabla\vec{v}\|_{L^2}\|\underline{w}\|_{L^2} \\ &+ \|\eta\|_{L^\infty}\|g\|_{L^2}\|\underline{w}\|_{L^2} + C\|\eta\|_{L^\infty}\|\vec{w}_0\|_{H^{1/2}}\|v\|_{L^4}\|\underline{w}\|_{L^2} \\ &+ C\kappa\|\vec{w}_0\|_{H^{1/2}}\|\nabla\underline{w}\|_{L^2} + 4\mu_r C\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^2}. \end{aligned} \tag{9}$$

3.2. **Problem for u.**

Our goal is to obtain $\vec{u} \in V$ solving (11) as a fixed point of the operator $\mathcal{A}$, to be defined below. To this end we shall apply the Leray-Schauder principle, which requires the operator $\mathcal{A}$ to be completely continuous and also that every possible solutions of $\vec{u} = \lambda\mathcal{A}\vec{u}$, $\lambda \in [0, 1]$, are uniformly bounded, see [11, 9].

Define $\mathcal{A} : V \to V$ as follows: for $\vec{u} \in V$, let $\underline{w} \in H_0^1(\Omega)$ denote the solution of problem (A) corresponding to $\vec{v} = \vec{u} + \vec{a}$ and consider $\mathcal{A}\vec{u}$ given by the identity

$$\begin{aligned}\sigma \int \nabla(\mathcal{A}\vec{u}) \cdot \nabla\varphi \, dx = &\int \eta(N[\vec{u}+\vec{a}])[(\vec{u}\cdot\nabla)\varphi\cdot\vec{u} + (\vec{u}\cdot\nabla)\varphi\cdot\vec{a} + (\vec{a}\cdot\nabla)\varphi\cdot\vec{u}]\, dx \\ &- \int \Big(2\mu_r \nabla^\perp \underline{w} - \eta(N[\vec{u}+\vec{a}])f\Big)\cdot\varphi\, dx + \int F\cdot\varphi\, dx,\end{aligned} \tag{10}$$

for all $\varphi \in V$. The well-definiteness of $\mathcal{A}$ follows from Riesz theorem. Indeed, the rigth-hand side of (10) is a continuous form on V owing to the properties of $\vec{a}$, cf. (5), and some elementary estimates.

Now we check $\mathcal{A}$ is completely continuous. For $\vec{u}^i \in V, i = 1,2$ let us define $\vec{v}^i = \vec{u}^i + \vec{a} \in \mathbf{H}$ and consider $\underline{w}^i \in H_0^1(\Omega)$ the solutions of problem (A) corresponding to $\vec{v}^i$. Then for $\varphi \in V,\ i = 1,2,$

$$\begin{aligned}\sigma \int \nabla(\mathcal{A}\vec{u}^i) \cdot \nabla\varphi \, dx = &\int \eta(N[\vec{u}^i+\vec{a}])(\vec{u}^i\cdot\nabla)\varphi\cdot\vec{u}^i\, dx \\ &+ \int \eta(N[\vec{u}^i+\vec{a}])[(\vec{u}^i\cdot\nabla)\varphi\cdot\vec{a} + (\vec{a}\cdot\nabla)\varphi\cdot\vec{u}^i]\, dx \\ &- \int \Big(2\mu_r \nabla^\perp \underline{w}^i - \eta(N[\vec{u}^i+\vec{a}])f\Big)\cdot\varphi\, dx + \int F_i\cdot\varphi\, dx,\end{aligned}$$

with $F_i = F(\eta(N(\vec{u}^i + \vec{a}), \vec{a}, b)$.

Subtracting these two identities we get

$$\sigma \int \nabla(\mathcal{A}\vec{u}^2 - \mathcal{A}\vec{u}^1)\cdot\nabla\varphi\, dx = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{11}$$

for

$$\begin{aligned}
I_1 &= \int \Big(\eta(N[\vec{u}^2+\vec{a}])(u^2\cdot\nabla)\varphi\cdot\vec{u}^2 - \eta(N[\vec{u}^1+\vec{a}])(\vec{u}^1\cdot\nabla)\varphi\cdot\vec{u}^1\Big)\, dx \\
I_2 &= \int \Big(\eta(N[\vec{u}^2+\vec{a}])(\vec{u}^2\cdot\nabla)\varphi\cdot\vec{a} - \eta(N[\vec{u}^1+\vec{a}])(\vec{u}^1\cdot\nabla)\varphi\cdot\vec{a}\Big)\, dx \\
I_3 &= \int \Big(\eta(N[\vec{u}^2+\vec{a}])(\vec{a}\cdot\nabla)\varphi\cdot\vec{u}^2 - \eta(N[\vec{u}^1+\vec{a}])(\vec{a}\cdot\nabla)\varphi\cdot\vec{u}^1\Big)\, dx \\
I_4 &= -\int 2\mu_r \nabla^\perp(\underline{w}^2 - \underline{w}^1)\cdot\varphi\, dx \\
I_5 &= +\int \big[\eta(N[\vec{u}^2+\vec{a}]) - \eta(N[\vec{u}^1+\vec{a}]))\big]\vec{f}\cdot\varphi\, dx \\
I_6 &= -\int \Big[\eta(N[\vec{u}^2+\vec{a}])) - \eta(N[\vec{u}^1+\vec{a}]))\Big](\vec{a}\cdot\nabla)\vec{a}\cdot\varphi\, dx.
\end{aligned}$$

We now show it is possible to bound each of the $I_k,\ k=1,\dots,6$, by a constant times $\|\vec{u}^2-\vec{u}^1\|_{L^4}$, proceeding as follows. First notice that

$$\begin{aligned}
I_1 &= \int \Big(\eta(N[\vec{u}^2+\vec{a}]) - \eta(N[\vec{u}^1+\vec{a}])\Big)(\vec{u}^2\cdot\nabla)\varphi\cdot\vec{u}^2\,dx \\
&\quad + \int \eta(N[\vec{u}^1+\vec{a}])\Big((\vec{u}^2\cdot\nabla)\varphi\cdot\vec{u}^2 - (\vec{u}^1\cdot\nabla)\varphi\cdot\vec{u}^1\Big)\,dx \\
&= \int \Big(\eta(N[\vec{u}^2+\vec{a}]) - \eta(N[\vec{u}^1+\vec{a}])\Big)(\vec{u}^2\cdot\nabla)\varphi\cdot\vec{u}^2\,dx \\
&\quad + \int \eta(N[\vec{u}^1+\vec{a}])\Big\{[(\vec{u}^2-\vec{u}^1)\cdot\nabla]\varphi\cdot\vec{u}^2 + (\vec{u}^1\cdot\nabla)\varphi\cdot(\vec{u}^2-\vec{u}^1)\Big\}\,dx = I_{11}+I_{12}.
\end{aligned}$$

Hölder, Young and imbedding inequalities imply

$$\begin{aligned}
|I_{11}| &\le \|\eta\|_{C^\alpha}\int |N(\vec{u}^2-\vec{u}^1)|^\alpha \Big|\sum_{j,k} u_k^2\partial_{x_k}\varphi_j u_j^2\Big|\,dx \\
&\le \|\eta\|_{C^\alpha}\sum_{j,k}\Big(\int(\partial_{x_k}\varphi_j)^2\,dx\Big)^{1/2}\Big(\sum_{j,k}\int |N(\vec{u}^2-\vec{u}^1)|^{2\alpha}[u_k^2u_j^2]^2\,dx\Big)^{1/2} \\
&\le C\|\eta\|_{C^\alpha}\|\nabla\varphi\|_{L^2}\|N(\vec{u}^2-\vec{u}^1)\|^\alpha_{W^{1,4\alpha}}\|\vec{u}^2\|^2_{L^8} \\
&= C\|\eta\|_{C^\alpha}\|\nabla\varphi\|_{L^2}\|\vec{u}^2-\vec{u}^1\|^\alpha_{L^{4\alpha}}\|\vec{u}^2\|^2_{L^8}, \\
|I_{12}| &\le \|\eta\|_{L^\infty}\|\nabla\varphi\|_{L^2}(\|\vec{u}^2\|_{L^4}+\|\vec{u}^1\|_{L^4})\|\vec{u}^2-\vec{u}^1\|_{L^4}.
\end{aligned}$$

Hence

$$\begin{aligned}
|I_1| &\le C\|\eta\|_{C^\alpha}\|\nabla\varphi\|_{L^2}\|\vec{u}^2-\vec{u}^1\|^\alpha_{L^{4\alpha}}\|\vec{u}^2\|^2_{L^8} \\
&\qquad + \|\eta\|_{L^\infty}\|\nabla\varphi\|_{L^2}(\|\vec{u}^2\|_{L^4}+\|\vec{u}^1\|_{L^4})\|\vec{u}^2-\vec{u}^1\|_{L^4}.
\end{aligned}\tag{12}$$

Likewise we get the bounds

$$\begin{aligned}
|I_2| &\le \|\eta\|_{L^\infty}\|\nabla\varphi\|_{L^2}\|\vec{u}^2-\vec{u}^1\|_{L^4}\|\vec{a}\|_{L^4} \\
&\qquad + C\|\eta\|_{C^\alpha}\|\vec{u}^2\|_{L^8}\|\vec{a}\|_{L^8}\|\nabla\varphi\|_{L^2}\|\vec{u}^2-\vec{u}^1\|^\alpha_{L^{4\alpha}} \\
|I_3| &\le \|\eta\|_{L^\infty}\|\nabla\varphi\|_{L^2}\|\vec{u}^2-\vec{u}^1\|_{L^4}\|\vec{a}\|_{L^4} \\
&\qquad + C\|\eta\|_{C^\alpha}\|\vec{u}^1\|_{L^8}\|\vec{a}\|_{L^8}\|\nabla\varphi\|_{L^2}\|\vec{u}^2-\vec{u}^1\|^\alpha_{L^{4\alpha}} \\
|I_5| &\le \|\eta\|_{C^\alpha}\|\vec{u}^2-\vec{u}^1\|^\alpha_{L^{4\alpha}}\|\vec{f}\|_{L^2}\|\varphi\|_{L^4} \\
|I_6| &\le C\|\eta\|_{C^\alpha}\|\vec{a}\|^2_{L^8}\|\nabla\varphi\|_{L^2}\|\vec{u}^2-\vec{u}^1\|^\alpha_{L^{4\alpha}}.
\end{aligned}\tag{13}$$

The term I_4 requires estimating $\|\underline{w}^2-\underline{w}^1\|_{H^1}$ in terms of $\|\vec{u}^2-\vec{u}^1\|_{L^4}$, a task we now perform. As $\underline{w}^i,\ i=1,2$ solve problem (A) we find that the following identity

$$\int \kappa\nabla(\underline{w}^2-\underline{w}^1)\cdot\nabla\psi + 4\mu_r(\underline{w}^2-\underline{w}^1)\psi\,dx = J_1+J_2+J_3,$$

holds for all $\psi \in H_0^1(\Omega)$, where

$$J_1 = \int \Big(\eta(N[\vec{u}^2+\vec{a}])(\vec{u}^2+\vec{a})\cdot\nabla\underline{w}^2 - \eta(N[\vec{u}^1+\vec{a}])(\vec{u}^1+\vec{a})\cdot\nabla\underline{w}^1\Big)\psi\,dx$$

$$J_2 = \int \Big(2\mu_r\nabla\times(\vec{u}^2-\vec{u}^1) + \big[\eta(N[\vec{u}^2+\vec{a}])(\vec{u}^2+\vec{a}) - \eta(N[\vec{u}^1+\vec{a}])(\vec{u}^1+\vec{a})\big]g\Big)\psi\,dx$$

$$J_3 = \int \Big(\eta(N[\vec{u}^2+\vec{a}])\vec{u}^2 - \eta(N[\vec{u}^1+\vec{a}])\vec{u}^1\Big)\cdot\nabla b\psi\,dx.$$

By arguing as in the estimations (12)-(13) we find

$$|J_1| \le C\Big(\|\eta\|_{C^\alpha}\|\vec{u}^2-\vec{u}^1\|_{L^{4\alpha}}^\alpha\|\vec{u}^1+\vec{a}\|_{L^8} + \|\eta\|_{L^\infty}\|\vec{u}^2-\vec{u}^1\|_{L^4}\Big)\|\nabla\underline{w}^2\|_{L^2}\|\psi\|_{H^1}$$

$$|J_2| \le C\Big(2\mu_r\mathrm{meas}(\Omega)^{1/4}\|\vec{u}^2-\vec{u}^1\|_{L^4} + \|\eta\|_{C^\alpha}\|g\|_{L^2}\|\vec{u}^2-\vec{u}^1\|_{L^{4\alpha}}^\alpha\Big)\|\psi\|_{H^1}$$

$$|J_3| \le C\Big(\|\eta\|_{C^\alpha}\|\vec{u}^2-\vec{u}^1\|_{L^{4\alpha}}^\alpha\|\vec{u}^1+\vec{a}\|_{L^8} + \|\eta\|_{L^\infty}\|\vec{u}^2-\vec{u}^1\|_{L^4}\Big)\|\nabla b\|_{L^2}\|\psi\|_{H^1}.$$

Taking $\psi = \underline{w}^2 - \underline{w}^1$ and invoking the boundedness of Ω, we get

$$\|\underline{w}^2-\underline{w}^1\|_{H^1} \le C\Big(\|\nabla\underline{w}^2\|_{L^2} + \|\nabla b\|_{L^2}\Big)\Big\{\|\eta\|_{C^\alpha}\|\vec{u}^1+\vec{a}\|_{L^8}\|\vec{u}^2-\vec{u}^1\|_{L^4}^\alpha + \|\eta\|_{L^\infty}\|\vec{u}^2-\vec{u}^1\|_{L^4}\Big\} + (2\mu_r\mathrm{meas}(\Omega)^{1/4} + \|\eta\|_{C^\alpha}\|g\|_{L^2})\|\vec{u}^2-\vec{u}^1\|_{L^4},$$

for some constant $C>0$ depending on κ,μ_r,Ω. At last, bearing in mind the triangle inequality, we find

$$\begin{aligned}|I_4| \le{}& 2\mu_r C\Big(\|\nabla\underline{w}^2\|_{L^2} + \|\nabla b\|_{L^2}\Big)\|\eta\|_{C^\alpha}(\|\vec{u}^1\|_{L^8} + \|\vec{a}\|_{L^8})\|\vec{u}^2-\vec{u}^1\|_{L^4}^\alpha\|\varphi\|_{H^1}\\ &+ 2\mu_r C(2\mu_r\mathrm{meas}(\Omega)^{1/4} + \|\eta\|_{C^\alpha}\|g\|_{L^2})\|\vec{u}^2-\vec{u}^1\|_{L^4}\|\varphi\|_{H^1}.\end{aligned} \tag{14}$$

Collecting inequalities (12)-(14) for $\varphi = \mathcal{A}\vec{u}^2 - \mathcal{A}\vec{u}^1$, together with $\|\vec{u}^2-\vec{u}^1\|_{L^4}\le 1$ and in view of (4) and (5), we conclude

$$\begin{aligned}\sigma\|\nabla(\mathcal{A}\vec{u}^2-\mathcal{A}\vec{u}^1)\|_{H^1} \le \|\vec{u}^2-\vec{u}^1\|_{L^4}^\alpha \times \Big\{&C\|\eta\|_{C^\alpha}\Big[\|\vec{u}^2\|_{L^8}(\|\vec{u}^2\|_{L^8} + C\|\vec{v}_0\|_{H^{1/2}})\\ &+ C\|\vec{v}_0\|_{H^{1/2}}(\|\vec{u}^1\|_{L^8} + C\|\vec{v}_0\|_{H^{1/2}}) + \|\vec{f}\|_{L^2} + \|g\|_{L^2}\\ &+ (\|\vec{u}^1\|_{L^8} + C\|\vec{v}_0\|_{H^{1/2}})(\|\nabla\underline{w}^2\|_{L^2} + C\|\vec{w}_0\|_{H^{1/2}})\Big]\\ &+ \|\eta\|_{L^\infty}\Big[\|\vec{u}^2\|_{L^4} + \|\vec{u}^1\|_{L^4} + 2C\|\vec{v}_0\|_{H^{1/2}}\\ &+ (\|\vec{u}^1\|_{L^8} + C\|\vec{v}_0\|_{H^{1/2}})(\|\nabla\underline{w}^2\|_{L^2} + C\|\vec{w}_0\|_{H^{1/2}})\\ &+ 2\mu_r\mathrm{meas}(\Omega)^{1/4}\Big]\Big\}.\end{aligned}$$

Thus, from the compactness of the imbedding $H^1 \hookrightarrow L^4$ and the inequality above, a weakly convergent sequence in V is mapped by $\mathcal{A}$ into a strongly convergent sequence in L^4. It remains to show that all possible solutions of the equation $\vec{u} = \lambda\mathcal{A}\vec{u}, \lambda\in[0,1]$, are

uniformly bounded. As $\lambda = 0$ implies $\vec{u} = 0$, we suppose $\lambda > 0$ and replace $\mathcal{A}\vec{u} = \vec{u}/\lambda$, $\varphi = \vec{u}$ in the equation (10). Performing estimations similar to the previous we find

$$\frac{\sigma}{\lambda}\|\nabla\vec{u}\|_{L^2}^2 \le \|\eta\|_{L^\infty}\left|\int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{a}\,dx\right| + 2\mu_r\big(\|\underline{w}\|_{L^2}+\|b\|_{L^2}\big)\|\nabla\vec{u}\|_{L^2} + \|\eta\|_{L^\infty}\|\vec{f}\|_{L^2}\|\vec{u}\|_{L^2}. \tag{15}$$

Notice that the term $2\mu_r\|\underline{w}\|_{L^2}\|\nabla\vec{u}\|_{L^2}$ itself is another quadratic term in $\vec{u}$ which we handle as follows. In view of (9), with $\vec{v} = \vec{u} + \vec{a}$, we find

$$\begin{aligned}\kappa\|\nabla\underline{w}\|_{L^2}^2 + 4\mu_r\|\underline{w}\|_{L^2}^2 &\le 2\mu_r\|\nabla(\vec{u}+\vec{a})\|_{L^2}\|\underline{w}\|_{L^2} + \|\eta\|_{L^\infty}\|g\|_{L^2}\|\underline{w}\|_{L^2}\\ &\quad + C\|\eta\|_{L^\infty}\|\vec{w}_0\|_{H^{1/2}}\|\vec{u}+\vec{a}\|_{L^4}\|\underline{w}\|_{L^4} + C\kappa\|\vec{w}_0\|_{H^{1/2}}\|\nabla\underline{w}\|_{L^2}\\ &\quad + 4\mu_r\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^2}.\end{aligned} \tag{16}$$

Aided by Young and Hölder inequalities we get, by summing up (15) and (16),

$$\begin{aligned}\frac{\mu+\mu_r}{\lambda}\|\nabla\vec{u}\|_{L^2}^2 + \kappa\|\nabla\underline{w}\|_{L^2}^2 &\le \|\eta\|_{L^\infty}\left|\int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{a}\,dx\right|\\ &\quad + \mu_r\|\nabla\vec{u}\|_{L^2}^2 + C\|\eta\|_{L^\infty}\|\vec{u}\|_{L^4}\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^4}\\ &\quad + 2\mu_r\|b\|_{L^2}\|\nabla\vec{u}\|_{L^2} + \|\eta\|_{L^\infty}\|\vec{f}\|_{L^2}\|\vec{u}\|_{L^2}\\ &\quad + 2\mu_r\|\nabla\vec{a}\|_{L^2}\|\underline{w}\|_{L^2} + \|\eta\|_{L^\infty}\|g\|_{L^2}\|\underline{w}\|_{L^2}\\ &\quad + C\|\eta\|_{L^\infty}\|\vec{a}\|_{L^4}\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^4} + C\kappa\|\nabla\underline{w}\|_{L^2}\|w_0\|_{H^{1/2}}\\ &\quad + 4\mu_r\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^2}.\end{aligned}$$

Next, requiring $\delta > 0$ in (5), to be such that $\delta\|\eta\|_{L^\infty} < \mu/2$ and estimating

$$\|\eta\|_{L^\infty}\|\vec{u}\|_{L^4}\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^4} \le \frac{C}{2}\|\eta\|_{L^\infty}\|\vec{w}_0\|_{H^{1/2}}\Big(\|\nabla\vec{u}\|_{L^2}^2 + \|\nabla\underline{w}\|_{L^2}^2\Big)$$

we ultimately get, using (4) and (5),

$$\begin{aligned}&\Big(\frac{\mu}{2\lambda} - \frac{C}{2}\|\eta\|_{L^\infty}\|\vec{w}_0\|_{H^{1/2}}\Big)\|\nabla\vec{u}\|_{L^2}^2 + \Big(\kappa - \frac{C}{2}\|\eta\|_{L^\infty}\|\vec{w}_0\|_{H^{1/2}}\Big)\|\nabla\underline{w}\|_{L^2}^2\\ &\qquad \le 2\mu_r C\|\vec{w}_0\|_{H^{1/2}}\|\nabla\vec{u}\|_{L^2} + \|\eta\|_{L^\infty}\|\vec{f}\|_{L^2}\|\vec{u}\|_{L^2} + 2\mu_r C\|\vec{v}_0\|_{H^{1/2}}\|\underline{w}\|_{L^2}\\ &\qquad + \|\eta\|_{L^\infty}\|g\|_{L^2}\|\underline{w}\|_{L^2} + C\|\eta\|_{L^\infty}\|\vec{v}_0\|_{H^{1/2}}\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^4}\\ &\qquad + C\kappa\|\nabla\underline{w}\|_{L^2}\|\vec{w}_0\|_{H^{1/2}} + 4\mu_r C\|\vec{w}_0\|_{H^{1/2}}\|\underline{w}\|_{L^2}.\end{aligned}$$

Therefore, demanding μ and κ to be large enough so that

$$\min\big\{\mu, 2\kappa\big\} > C\|\eta\|_{L^\infty}\|\vec{w}_0\|_{H^{1/2}},$$

we may conclude the uniform boundedness on the norms of all possible solutions of $\vec{u} = \lambda\mathcal{A}\vec{u},\ \lambda \in [0,1]$, with respect to the parameter λ. From this and previous steps

we conclude that the Leray-Schauder principle applies and that problem (6) has a weak solution. □

4. Concluding remarks

We have tacitly assumed $\partial\Omega$ to consist of a single component, that is, $\Omega \subset \mathbb{R}^2$ to be a simply connected open set. A clue on how the arguments should be modified to cope with a non-simply connected Ω may be found in [5].

The presence of the quadratic term brought up by the equation $(1)_4$, for the rotational field w, required us to demand the viscosities μ, κ to be sufficiently large compared to data and this somewhat contrasts with previous results by Frolov [5] and Santos [13].

As shown in [5, cf. Theorems 2 and 3], also [11, Chapter 2], the regularity of $\rho, \vec{v}, w, p$ may be improved by increasing those of $\partial\Omega, \rho_0, \vec{v}_0, \vec{w}_0, \vec{f}$ and g. An usual bootstrap argument should combine regularity results for the Stokes problem and Nečas' results on strongly elliptic systems of second order [12].

We also notice that the uniqueness of the above solution deserves further investigation. Indeed we could not benefit from neither the steady-state continuity equation $(1)_2$ nor the particular form of the density to derive the required estimates.

References

[1] Antontsev, S.N., Kazhikhov, A.V., Monakhov, V.N.: Boundary value problems in mechanics of nonhomogeneous fluids, *Studies in Mathematics and its Applications*, vol. 22. North-Holland Publishing Co., Amsterdam (1990). Translated from the Russian

[2] Boldrini, J.L., Rojas-Medar, M.A., Fernández-Cara, E.: Semi-galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids. J. Math. Pures Appl. (9) **82**(11), 1499–1525 (2003). DOI 10.1016/j.matpur.2003.09.005. URL `http://dx.doi.org/10.1016/j.matpur.2003.09.005`

[3] Conca, C., Gormaz, R., Ortega, E., Rojas, M.: Stud. math. appl. In: Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIV (Paris, 1997/1998), vol. 31, chap. Existence and uniqueness of a strong solution for nonhomogeneous micropolar fluids, pp. 213–241. North-Holland, Amsterdam (2002)

[4] Eringen, A.C.: Theory of micropolar fluids. J. Math. Mech. **16**, 1–18 (1966)

[5] Frolov, N.N.: On the solvability of a boundary value problem of the motion of a nonhomogeneous fluid. Mat. Zametki **53**(6), 130–140 (1993). DOI 10.1007/BF01212604. URL `http://dx.doi.org/10.1007/BF01212604`

[6] Guillén-González, F., Poblete-Cantellano, M., Rojas-Medar, M.: On the variational inequalities related to viscous density-dependent incompressible fluids. Annali dell'Universita di Ferrara **56**(1), 163–180 (2010). URL `http://dx.doi.org/10.1007/s11565-009-0087-z`. 10.1007/s11565-009-0087-z

[7] Heywood, J.G., Rannacher, R., Turek, S.: Artificial boundaries and flux and pressure conditions for the incompressible navier-stokes equations. Internat. J. Numer. Methods Fluids **22**(5), 325–352

(1996). DOI 10.1002/(SICI)1097-0363(19960315)22:5⟨325::AID-FLD307⟩3.0.CO;2-Y. URL http://dx.doi.org/10.1002/(SICI)1097-0363(19960315)22:5<325::AID-FLD307>3.0.CO;2-Y

[8] Illarionov, A.A.: Optimal boundary control of the steady flow of a viscous nonhomogeneous incompressible fluid. Mat. Notes **69**(5-6), 614–624 (2001). DOI 10.1023/A:1010297424324. URL http://dx.doi.org/10.1023/A:1010297424324

[9] Ladyzhenskaya, O.A.: The mathematical theory of viscous incompressible flow. Revised English edition. Translated from the Russian by Richard A. Silverman. Gordon and Breach Science Publishers, New York (1963)

[10] Lions, P.L.: Mathematical topics in fluid mechanics. Vol. 1, *Oxford Lecture Series in Mathematics and its Applications*, vol. 3. The Clarendon Press Oxford University Press, New York (1996). Incompressible models, Oxford Science Publications

[11] Łukaszewicz, G.: Micropolar fluids. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston Inc., Boston, MA (1999). Theory and applications

[12] Nečas, J.: L'application de l'égalité de rellich sur les systèmes elliptiques du deuxième ordre. J. Math. Pures Appl. (9) **44**, 133–147 (1965)

[13] Santos, M.M.: Stationary solution of the navier-stokes equations in a 2d bounded domain for incompressible flow with discontinuous density. Z. Angew. Math. Phys. **53**(4), 661–675 (2002). DOI 10.1007/s00033-002-8175-2. URL http://dx.doi.org/10.1007/s00033-002-8175-2

[14] Temam, R.: Navier-Stokes equations. AMS Chelsea Publishing, Providence, RI (2001). Theory and numerical analysis, Reprint of the 1984 edition

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